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On the non-integrability of Yang–Mills potentials

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Abstract. The non-integrability of three Hamiltonian systems obtained by simplifications of the Yang-Mills classical field theory is proved by application of two theorems of Yoshida. It is also shown that for the two four-degree of freedom systems there exists a set of three independent integrals of motion in involution.

1. Introduction

It is known (Matinyan *et al* 1981, Asatryan and Savvidy 1983, Savvidy 1984) that the equations of motion of the SU(2) Yang-Mills classical field theory, under the assumption that the potentials depend only on time and under a more restrictive ansatz, yield the classical system of equations

$$\ddot{x} = -xy^2 \tag{1a}$$

$$\ddot{y} = -yx^2 \tag{1b}$$

hereafter named system A, which can be derived from the Hamiltonian

$$H_1 = \tfrac{1}{2}(p_x^2 + p_y^2) + \tfrac{1}{2}x^2y^2. \tag{2}$$

In the above references, it is conjectured that system A is a strongly chaotic non-integrable system and this was confirmed by numerical investigation on this system by Carnegie and Percival (1984), where it was suggested that no regular regions of motion exist on the surface of section and the motion is always irregular, except for a measure zero set of unstable periodic orbits. In a recent paper, Villarroel (1988) obtained the classical system of four degrees of freedom by making a less restrictive ansatz on the potentials:

$$\ddot{x} + g^2w(xw - yz) = 0 \tag{3a}$$

$$\ddot{y} - g^2z(xw - yz) = 0 \tag{3b}$$

$$\ddot{z} - g^2y(xw - yz) = 0 \tag{3c}$$

$$\ddot{w} + g^2x(xw - yz) = 0 \tag{3d}$$

hereafter named system B, which can be derived from the Hamiltonian

$$H_2 = \tfrac{1}{2}(p_x^2 + p_y^2 + p_z^2 + p_w^2) + (g^2/2)(xw - yz)^2 \tag{4}$$

where g is a coupling constant.

An attempt to prove non-integrability of system B by the Lobatchewsky-Hadamard theorem was unsuccessful, since it provided a non-negative scalar curvature of the

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corresponding local Riemannian manifold of the flow and non-integrability was then conjectured on the fact that the Painlevé analysis of this system provided a pair of complex resonances, indicating the presence of an algebraic branch point.

An extension to the case of a SU(2) gauge system with spontaneous symmetry breaking yielded the system described by the Hamiltonian

$$H_3=\tfrac{1}{2}(p_x^2+p_y^2+p_z^2+p_w^2)+(g^2/2)(xw-yz)^2+(g^2K^2/4)(x^2+y^2+z^2+w^2) \tag{5}$$

hereafter called system C, where K is a constant parameter. This system is integrable in the limit $K\to\infty$ and its periodic solutions were studied via perturbation theory by using the action-angle variables of the integrable part.

In this paper, non-integrability of system A is proved by a direct application of a theorem by Yoshida (1987) in § 2. In § 3, the integrability properties of system B are investigated. By the application of two successive point transformations, it is shown that this system possesses two independent integrals of motion in involution, in addition to the Hamiltonian, which can be obtained as generalised momenta, conjugate to ignorable coordinates in the transformed system. By using these integrals, system B can be reduced to a system of two degrees of freedom with a non-homogeneous potential. Non-integrability of this latter system is then proved by application of another theorem of Yoshida (1988). Finally, in § 4 it is shown that non-integrability of system C can also be proved by exactly the same procedure.

2. Non-integrability of system A

Yoshida's (1987) theorem, which can provide a sufficient condition for non-integrability for a two degrees of freedom Hamiltonian of the form

$$H=\tfrac{1}{2}(p_1^2+p_2^2)+V(q_1,q_2) \tag{6}$$

with a homogeneous potential $V(q_1,q_2)$ of degree k ($\neq 0, \pm 2$) has been proved by applying Ziglin's theorem (Ziglin 1983a, b) to the straight-line solutions of system (6) and, for the case $k\geqslant 3$, it is stated as follows.

Let c_1, c_2 be a solution of the algebraic system of equations

$$\frac{\partial V}{\partial q_i}(c_1,c_2)=c_i \qquad i=1,2.$$

The quantity λ, which is called the 'integrability coefficient', is defined by the equation

$$\lambda=\sum_{i=1}^{2}\frac{\partial^2 V}{\partial q_i^2}(c_1,c_2)-(k-1). \tag{7}$$

If λ lies in the non-integrability region

$$S_k=\{\lambda<0,\ 1<\lambda<k-1,\ k+2<\lambda<3k-2,\ldots\}$$

then no additional integral of motion for system (6) exists and consequently the system is non-integrable.

By applying the above theorem to system A, we obtain $c_1^2=c_2^2=1$ and $\lambda=-1<0$, i.e. $\lambda\in S_4$ which proves that system A is non-integrable.

3. Integrals of motion and non-integrability for system B

If we perform the following orthogonal point transformation from (x, y, z, w) to $(\xi_1, \xi_2, \xi_3, \xi_4)$

$$\xi_1 = 2^{-1/2}(x+w) \qquad \xi_2 = 2^{-1/2}(x-w)$$
$$\xi_3 = 2^{-1/2}(y+z) \qquad \xi_4 = 2^{-1/2}(y-z) \tag{8}$$

the Hamiltonian (4) of system B takes the form

$$H_2 = \tfrac{1}{2}(P_1^2 + P_2^2 + P_3^2 + P_4^2) + \tfrac{1}{8}g^2(\xi_1^2 - \xi_2^2 + \xi_3^2 - \xi_4^2)^2. \tag{9}$$

Then we transform to polar coordinates on the (ξ_1, ξ_3) and (ξ_2, ξ_4) planes by the point transformation

$$\xi_1 = r_1 \cos\theta_1 \qquad \xi_2 = r_2 \cos\theta_2$$
$$\xi_3 = r_1 \sin\theta_1 \qquad \xi_4 = r_2 \sin\theta_2 \tag{10}$$

so the Hamiltonian (9) becomes

$$H_2 = \tfrac{1}{2}(p_{r_1}^2 + p_{r_2}^2 + p_{\theta_1}^2 r_1^{-2} + p_{\theta_2}^2 r_2^{-2}) + \tfrac{1}{8}g^2(r_1^2 - r_2^2)^2. \tag{11}$$

The angles θ_1, θ_2 are now ignorable coordinates, so the corresponding momenta are integrals of motion:

$$p_{\theta_1} = \dot{\xi}_3\xi_1 - \dot{\xi}_1\xi_3 = \text{constant} = a$$
$$p_{\theta_2} = \dot{\xi}_4\xi_2 - \dot{\xi}_2\xi_4 = \text{constant} = b. \tag{12}$$

By going back to the initial set of coordinates (x, y, z, w), we may rewrite the two integrals as follows:

$$I_1 = p_{\theta_1} + p_{\theta_2} = (x\dot{y} - \dot{x}y) + (w\dot{z} - z\dot{w}) = \text{constant} = C_1$$
$$I_2 = p_{\theta_1} - p_{\theta_2} = (x\dot{z} - \dot{x}z) + (w\dot{y} - y\dot{w}) = \text{constant} = C_2 \tag{13}$$

which are two independent additional integrals of motion for system B. It is easy to check that $[I_1, I_2] = 0$, so the integrals I_1 and I_2 are in involution.

In Villarroel (1988), the value of C_1 has been selected equal to zero and the corresponding relation:

$$I_1 = p_{\theta_1} + p_{\theta_2} = 0 \tag{14}$$

represented Gauss' law for the initial Yang-Mills system and was treated as a non-holonomous constraint. Since I_1 is an integral of motion of system B, Gauss' law corresponds to a fixed value of the constant C_1 and is fulfilled on the corresponding level surface in the phase space of the system.

Now, by using the integrals of motion (12), we are able to make a reduction to a two degrees of freedom system

$$H_2' = \tfrac{1}{2}(P_X^2 + P_Y^2 + a^2X^{-2} + b^2Y^{-2}) + \tfrac{1}{8}g^2(X^2 - Y^2)^2 \tag{15}$$

where X and Y stand for r_1 and r_2, respectively, and a and b are arbitrary constant parameters. If a solution of (15) is known, the corresponding solution for the original four degrees of freedom system can be completed by integrating the relations

$$\dot{\theta}_1 = a r_1^{-2}$$
$$\dot{\theta}_2 = b r_2^{-2}.$$

If, moreover, non-integrability is proved for the reduced system, even for some particular value of an integral of motion, then it is obvious that the complete four degrees of freedom system is non-integrable. If we select in (15), $a=0$, then H_2' becomes

$$H_2^* = \tfrac{1}{2}(P_X^2 + P_Y^2) + V^*(X, Y). \tag{16}$$

The corresponding potential V^* is of the form

$$V^* = V_{-2} + V_4 \tag{17}$$

where

$$V_{-2} = b^2 Y^{-2}/2$$

and

$$V_4 = \tfrac{1}{8} g^2 (X^2 - Y^2)^2$$

are homogeneous functions of degree −2 and 4, respectively.

The theorem of Yoshida (1988) we are now going to apply is stated as follows.

Consider the Hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + V(q_1, q_2) \tag{18}$$

with a non-homogeneous potential V, which can be written as a finite sum of homogeneous parts V_k:

$$V = \sum_k V_k(q_1, q_2)$$

and assume that $q_1 = 0$ is a straight-line solution of the system, i.e.

$$\partial V / \partial q_1 = 0 \qquad \text{for } q_1 = 0.$$

Let $\lambda_{k_{\min}}$, $\lambda_{k_{\max}}$ be the integrability coefficients along the above solution for the lowest and highest parts $V_{k_{\min}}$, $V_{k_{\max}}$, respectively, and let $S_{k_{\min}}$, $S_{k_{\max}}$ be the corresponding non-integrability regions. If either $\lambda_{k_{\min}} \in S_{k_{\min}}$ or $\lambda_{k_{\max}} \in S_{k_{\max}}$ holds, then no additional integral for system (18) can exist and the system is non-integrable.

Note that $\partial V^*/\partial X = 0$ for $X = 0$, so for system (18) the desired straight-line solution does exist. The integrability coefficient λ_4 of the fourth-degree part can be found as described in § 2 and is $\lambda_4 = -1$. According to Yoshida's theorem, since $\lambda_4 \in S_4$, non-integrability of (16), and subsequently of system B, has been proved.

4. Non-integrability of system C

Non-integrability of this system can be proved by following exactly the same steps as with system B. By performing the transformations (8) and (10), the Hamiltonian (5) takes the form

$$H_3 = \tfrac{1}{2}(p_{r_1}^2 + p_{r_2}^2 + p_{\theta_1}^2 r_1^{-2} + p_{\theta_2}^2 r_2^{-2}) + \tfrac{1}{4} g^2 K^2 (r_1^2 + r_2^2) + \tfrac{1}{8} g^2 (r_1^2 - r_2^2)^2. \tag{19}$$

Again I_1 and I_2 (or p_{θ_1} and p_{θ_2}) are independent integrals of motion in involution and the reduced two-dimensional system, for $p_{\theta_1} = 0$, $p_{\theta_2} = b$, takes the form

$$H_3^* = \tfrac{1}{2}(P_X^2 + P_Y^2) + V^*(X, Y).$$

The corresponding potential is

$$V^* = V_{-2} + V_2 + V_4$$

where V_{-2} and V_4 are as in the preceding section, while

$$V_2 = \tfrac{1}{4}g^2K^2(X^2+Y^2).$$

The straight-line solution $X=0$ still exists and non-integrability of V_4 suffices for the non-integrability of system C.

5. Conclusions

The non-integrability of three Hamiltonian systems which represent simplified versions of the equations of the SU(2) Yang–Mills classical field theory is proved by making use of two theorems by Yoshida.

Moreover, it is shown that, in two of these systems with four degrees of freedom, two additional angular momentum-like integrals exist which form, with the Hamiltonian of the system, a set of three independent integrals in involution. These are not sufficient, however, for Liouville integrability and the above systems are non-integrable.

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